REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
UNCLASSIFIED	

DTIC
ELECTE
JUL 1 2 1989
S D

2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION / AVAILABILITY OF REPORT
N/A	Approved for public release;
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE	distribution unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S)	5. MONITORING ORGANIZATION REPORT NUMBER(S)
	AFOSR-TR- 89-0916

6a. NAME OF PERFORMING ORGANIZATION	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION
Stanford University		Air Force Office of Scientific Research

6c. ADDRESS (City, State, and ZIP Code)	7b. ADDRESS (City, State, and ZIP Code)
660 Arguello Way	Building 410
Encina Hall	Bolling AFB, DC 20332-6448
Stanford, CA 94305-6080	

8a. NAME OF FUNDING / SPONSORING ORGANIZATION	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
AFOSR	NM	AFOSR-83-0228

8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS			
	PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
Building 410	61102F	2304	A6	
Bolling AFB, DC 20332-6448				

11. TITLE (Include Security Classification)

Studies in Statistical Signal Processing

12. PERSONAL AUTHOR(S)

Prof. Thomas Kailath

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
FINAL	FROM 1 Jul 83 TO 30 Jun 86		

16. SUPPLEMENTARY NOTATION

17.	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The primary focus of this research was on the development of efficient and numerically stable algorithms for nonstationary signal processing. A secondary focus was to establish and broaden links with related disciplines, such as cascade filter synthesis, scattering theory, numerical linear algebra, and mathematical operator theory for the purpose of cross fertilization of ideas and techniques. These explorations have led to new results both in estimation theory and in these other fields, e.g., to new algorithms for triangular and QR factorization of structured matrices, new techniques for root location and stability testing, new recursions for orthogonal polynomials on the unit circle and the real line as well as on other curves, and new approaches to overcome singularities and ill-conditioning in the recursions.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL
Eytan Bar	(202) 767-4939	NM

DD Form 1473, JUN 86   Previous editions are obsolete.   SECURITY CLASSIFICATION OF THIS PAGE

AD-A210 054

STUDIES IN STATISTICAL SIGNAL PROCESSING

FINAL TECHNICAL REPORT ON CONTRACT AFOSR-83-0228

Professor Thomas Kailath

Information Systems Laboratory
Electrical Engineering Department
STANFORD UNIVERSITY, Stanford, CA-94305-4055

DTIC
COPY
INSPECTED
1

Accession For	
NTIS CRA&I	☑
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and/or Special
A-1	

STUDIES IN STATISTICAL SIGNAL PROCESSING

This is a final report on Contract AFOSR83-0228, for the period of July 1, 1986 and June 30, 1988. Section I provides a brief overview of our work, while the remaining sections describe in some detail our recent results on efficient factorization of structured matrices, polynomial zero-location, handling of singularities in the recursions, and recursive layer peeling.

SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND
DEGREES AWARDED DURING THIS REPORTING PERIOD:

Professor T. Kailath
Graduate Student: R. Ackner
Senior Research Associate: H. Lev-Ari

1. INTRODUCTION

The primary objective of our research is to develop efficient and numerically stable algorithms for nonstationary signal processing problems by understanding and exploiting special structures, both deterministic and stochastic, in the problems. We also strive to establish and broaden links with related disciplines, such as cascade filter synthesis, scattering theory, numerical linear algebra, and mathematical operator theory for the purpose of cross fertilization of ideas and techniques. These explorations have led to new results both in estimation theory and in these other fields, e.g., to new algorithms for triangular and QR factorization of structured matrices, new techniques for root location and stability testing, new recursions for orthogonal polynomials on the unit circle and the real line as well as on other curves, and new approaches to overcome singularities and ill-conditioning in the recursions.

For several years, the guiding principle in these studies has been the concept of generalized displacement structure (Lev-Ari and Kailath (1986)), which generalized and subsumed our earlier work on Toeplitz-oriented displacement structure (Kailath, Kung and Morf, (1979); see also Lev-Ari and Kailath (1984)). The same notion of

displacement structure has also emerged from the work of Heinig and Rost (1984,1987) of East Germany, whose approach and methodology are significantly different from ours. In particular, they focus only on the problem of inversion of structured matrices via algebraic methods, while our work has primarily addressed triangular factorization of such matrices, and our approach is based on a generating function characterization of matrices. In fact, in the recent Ph.D. research of J.Chun we were able to reduce the inversion problem for structured matrices to the factorization of certain block-matrices with structured blocks. This result confirms our earlier findings (in Lev-Ari(1983), Lev-Ari and Kailath (1984)) on the relation between efficient inversion and efficient factorization of structured matrices: only some of the structured matrices that admit an efficient factorization procedure can also be efficiently inverted.

The generating function approach, which was introduced in the Ph.D. research of Lev-Ari (1983), also suggests a natural geometric interpretation of the theory, which allows a number of other interesting developments, e.g., studies of various problems in system theory, such as minimal realization, Padé approximation, control design, and a variety of root distribution (stability) problems for polynomials. This approach also unveils the great flexibility in the computational details of the factorization procedure for structured matrices. In contrast, the approach taken by Heinig and Rost (and related work by Lerer, Tismenetsky, Shalom and several other mathematicians), leads to a single procedure for (the inversion of) every particular type of structured matrices. In particular, our generating function approach led us to recognize some classical root location procedures, such as the Schur-Cohn and the Routh-Hurwitz algorithms, as particular instances of our factorization procedure. Moreover, by exploiting the

flexibility in our prototype procedure we obtained new alternatives to these classical algorithms, with reduced computational requirements.

2. FACTORIZATION OF STRUCTURED MATRICES

Our early work on factorization and inversion of Toeplitz and close-to-Toeplitz matrices led us to the observation that for certain matrices the *displacement matrix*

$$\nabla_Z R := R - ZRZ^*, \quad Z = [\ \delta_{i,j}\ ]_{i,j=0}^{n}$$

has low rank. In particular, the displacement rank (i.e., the rank of $\nabla_Z R$) is 2 for both Toeplitz matrices and for their inverses. We have shown in previous work (largely supported by AFOSR) that the displacement concept is a key tool for, developing fast algorithms of many kinds, including factorization and inversion of Toeplitz and near-Toeplitz matrices, as well as fast (generalized Levinson and Schur) algorithms for solving linear systems with such coefficient matrices. Not surprisingly, these results led naturally to cascade orthogonal structures (generalized lattice filters) for the prediction of nonstationary processes (Lev-Ari and Kailath (1984)). We have also found that the same concept is tightly connected to the more general problem of cascade filter synthesis in network theory and digital filtering as well as to a variety of inverse scattering problems (some references are Rao and Kailath, (1984, 1985), Bruckstein and Kailath (1986)).

Later we extended the concept of displacement structure to a very broad family of structured matrices, including Hankel matrices and their inverses, sums of Toeplitz and Hankel matrices and several others (Lev-Ari and Kailath, (1986)). The generalized displacement of a matrix R, is defined as $d(Z,Z)R$ where

$$d(A,B)R := \sum_{k,l=0}^{N} d_{k,l} Z^k R (Z^*)^l , \qquad (1a)$$

and the asterisk $(*)$ denotes Hermitian transpose (complex conjugate for scalars). The generalized displacement operator $d(Z,Z)$ is characterized by a (Hermitian) displacement matrix J_d,

$$J_d := \{d_{k,l} ; 0 \le k,l \le N\} . \qquad (1b)$$

The previous notion of displacement corresponds to the particular displacement matrix

$$J_d = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} := J_T ,$$

while the displacement notion used by Heinig and Rost (1984) for Hankel matrices, corresponds to the displacement matrix

$$J_d = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} := J_R .$$

We have shown that efficient factorization procedures for a Hermitian matrix R can be formulated if, and only if, $d(Z,Z)R$ has low rank, and if the displacement matrix J_d has inertia $(1,1)$. This means that J_d must have the form

$$J_d = \phi\phi^* - \psi\psi^* \qquad (2)$$

where ϕ, ψ are column vectors of arbitrary length.

The concept of displacement structure and its properties are more conveniently described in terms of *generating functions*. The generating function of a matrix R is a power series in two complex variables, viz.,

$$R(z,w) := [1 \quad z \quad z^2 \ ...]R[1 \quad w \quad w^2 \ ...]^* \qquad (3)$$

The displacement $d(\mathbf{Z},\mathbf{Z})\mathbf{R}$ of a matrix has the generating function $d(z,w)R(z,w)$, where $d(z,w)$ is the generating function of the Hermitian matrix J_d , viz.,

$$d(z,w) = \sum_{k,l=0}^{N} d_{k,l}\ z^k (w^*)^l \tag{4}$$

Thus the generating function of a Hermitian matrix with a displacement structure has the form

$$R(z,w) = \frac{G(z)\ J\ G^*(w)}{d(z,w)} \tag{5}$$

where J is any constant nonsingular Hermitian matrix. The triple $\{d(z,w),G(z),J\}$ is called a generator of $R(z,w)$.

We have extended our previous work (Lev-Ari and Kailath (1984)) on efficient factorization of matrices with displacement structure to accommodate the generalized displacement $d(\mathbf{Z},\mathbf{Z})\mathbf{R}$, and we have shown (Lev-Ari and Kailath (1986), Lev-Ari, Bistritz and Kailath (1987)) that efficient factorization of $\mathbf{R}$ is possible if there exist matrix functions $\Theta_i(z)$ and constant matrices J_i $(J_0 := J)$ that satisfy the matrix equation

$$\Theta_i(z) J_{i+1} \Theta_i^*(w) \ = \ J_i - \frac{d(z,w)}{d(z,\zeta_i)d(\zeta_i,w)}\ J_i M_i J_i \tag{6a}$$

for arbitrary ζ_i , where

$$M_i \ := \ G_i^*(\zeta_i)R_i^{-1}(\zeta_i,\zeta_i)G_i(\zeta_i) = M_i^* \ . \tag{6b}$$

We have also shown that (6) has a solution if, and only if, J_i are all congruent to J and

$$d(z,w) = \phi(z)\phi^*(w) - \psi(z)\psi^*(w) \tag{7}$$

which is the generating function version of (2). The standard choice $\zeta_i = 0$ produces triangular factorizations; other choices are required in root-location and filter synthesis procedures (see, e.g., Deprettere and Dewilde (1980), Vaidyanathan and Mitra (1984)).

The (nonunique) solution of (6) is

$$\Theta_i(z) = \left\{ I - \frac{d(z,\tau_i)}{d(z,\zeta_i)d(\zeta_i,\tau_i)} \, J_i M_i \right\} U_i \tag{8a}$$

where U_i is any constant matrix such that

$$U_i \, J_{i+1} \, U_i^* = J_i \quad , \tag{8b}$$

and τ_i is any complex constant such that

$$d(\tau_i, \tau_i) = 0 \quad . \tag{8c}$$

The factorization of **R** is obtained via the recursion

$$(z - \zeta_i) G_{i+1}(z) = G_i(z)\Theta_i(z) \qquad i = 0,1,2, \ldots \tag{9}$$

where $\Theta_i(z)$ have the form (8). This algorithm requires $O(n^2)$ computations to factor a structured $n \times n$ matrix, in contrast to the conventional LDL^* algorithm which requires $O(n^3)$ operations to factor an arbitrary $n \times n$ matrix.

It should be observed that the formulation (8) allows a great flexibility in the selection of the scalars ζ_i, τ_i and the matrices U_i. The "universe of choice" has four distinct dimensions that determine the specific form of an efficient factorization procedure for a *given displacement function* $d(z,w)$:

(i) **Generator Type:** If $J = J_T$ we say that the *generator* $\{G(z), J\}$ and the corresponding recursion are of the *scattering-type*. This terminology is motivated by the observation that with $J = J_T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ we can rewrite (5) as

$$R(z,w) = \frac{u(z)u^*(w) - v(z)v^*(w)}{d(z,w)} \tag{10a}$$

and that

$$\mathbf{R} \geq 0 \iff \sup_{z \in \Omega_+} \left| \frac{v(z)}{u(z)} \right| \leq 1 . \tag{10b}$$

This means that the ratio $v(z)/u(z)$ can be interpreted as a (generalized) scattering function of a passive system. In particular, this ratio is a *continuous-time scattering-function* when $d(z,w) = z + w^*$ and a *discrete-time scattering function* when $d(z,w) = 1 - zw^*$. Similarly, if $J = J_I := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we say that the generator $\{G(z), J\}$ and the corresponding recursion are of the *immittance-type*. In this case (5) can be rewritten as

$$R(z,w) = \frac{f(z) g^*(w) + g(z) f^*(z)}{d(z,w)} \tag{11a}$$

which implies that

$$\mathbf{R} \geq 0 \iff \inf_{z \in \Omega_+} \left\{ \operatorname{Re} \frac{g(z)}{f(z)} \right\} \geq 0 . \tag{11b}$$

Thus the ratio $g(z)/f(z)$ is *positive real* in Ω_+, which for $d(z,w) = 1 - zw^*$ (respectively $z + w^*$) corresponds to a discrete-time (respectively continuous-time) *impedance/admittance* (= immittance) function. Other choices of J determine other types.

(ii) **Extraction Point:** The *extraction points* $\{\zeta_i\}$ can be either on the curve Ω_0, defined by $d(z,z) = 0$, or in one of the (disjoint) domains Ω_+, Ω_-. In most applications there is an obvious preference for *fixed, real-valued* extraction points, i.e., $\zeta_i = \zeta_0$ for all i, where ζ_0 is real. However, in other cases it may be necessary to use a different (complex-valued) ζ_i at each extraction step (see,

e.g., Vaidyanathan and Mitra (1984)). Finally, the choice of ζ_i affects the computational complexity of evaluating $G_i(\zeta_i)$; the best choices from this point of view are $\zeta_i = 0$ and (probably next best) $\zeta_i = \pm 1$.

(iii) **Recursion Depth:** The fundamental recursion (9) is a *two-term vector* recursion. This means that each step of the recursion relates variables indexed by i and by $i + 1$ (hence "two-term") and that there are several functions, combined into a vector $G_i(z)$ for each index point i. In other words, we can interpret (9) as a linear vector difference equation of the first order. Such equations can always be transformed into a single scalar linear difference equation of a higher order. In particular, if $G(z)$ has two elements (i.e., if J is a 2×2 matrix), then (9) can be transformed into a scalar linear difference equation of the second order, also known as a *three-term recursion*. For instance, the Schur-Cohn is a two-term recursion for a vector function of length two, while the Bistritz test (see Bistritz (1984)) is a three-term recursion for a scalar function.

(iv) **Recursion Parameters:** The scalars $\{\tau_i\}$ and the matrices $\{U_i\}$ can be chosen arbitrarily, subject to the constraints (8b), (8c). It seems, however, that minimization of computational complexity dictates specific (and, essentially, unique) choices for these parameters.

Our recent research has identified several choices that result in recursions whose computational requirements are less than those of known procedures.

In particular, we have constructed computationally-improved alternatives to the classical Schur (and Levinson) algorithm, and to the Schur-Cohn test, as follows:

(i)   Immir .nce-domain three-term equivalents of the Levinson and Schur recursions for quasi-Toeplitz matrices *with complex-valued elements* (Bistritz, Lev-Ari and Kailath (1987)). This extends our previous results for real-valued quasi-Toeplitz matrices (Bistritz, Lev-Ari and Kailath (1986)).

(ii)   An alternative to the Schur-Cohn algorithm for root-location with respect to the unit circle; the form of this new recursion is an exact equivalent of the Routh-Hurwitz algorithm (Lev-Ari (1987)).

On the theoretical level we have recently established the following fundamental result: for every $d(z,w)$ of the form (7) and for every (finite) matrix $\mathbf{R}$,

$$In \ \{d(\mathbf{Z},\mathbf{Z})\mathbf{R}\} \ = \ In \ \{d(\mathbf{Z},\mathbf{Z})[rev \ \mathbf{R}^{-1}]\} \tag{12}$$

where the reversed matrix $rev \ \mathbf{R}^{-1}$ is obtained by transposing $\mathbf{R}^{-1}$ with respect to the secondary diagonal. Thus the *reversal* of any matrix $\mathbf{Q}$ is formally defined as

$$rev \ \mathbf{Q} := \tilde{\mathbf{I}} \ \mathbf{Q}^T \ \tilde{\mathbf{I}} \tag{13a}$$

where the superscript T denotes the conventional (non-Hermitian) transpose, and

$$\tilde{\mathbf{I}} := \begin{bmatrix} & & & 1 \\ & 0 & \cdot & \\ & \cdot & & \\ & \cdot & 0 & \\ 1 & & & \end{bmatrix} \tag{13b}$$

The fundamental result (12) implies that $\mathbf{R}$ and $rev \ \mathbf{R}^{-1}$ have the same displacement structure. It does not tell us, however, how to relate the parametrizations of generators of $\mathbf{R}$ and of $rev \ \mathbf{R}^{-1}$. Such relations are known for Toeplitz matrices and give rise to the Levinson algorithm and the Gohberg-Semencul formula. They have been extended by Lev-Ari and Kailath (1984) to close-to-Toeplitz matrices, i.e.,

to matrices with displacement structure involving $d(z,w) = 1 - zw^*$.

The proof of the result (12) is based on several observations, including the following:

(i) Under the constraint (7), the generalized displacement $d(Z,Z)R$ is given by

$$d(Z,Z)R = \phi(Z) R \phi^*(Z) - \psi(Z) R \psi^*(Z)$$

(ii) We can assume without loss of generality that $\phi(z)$ and $\psi(z)$ do not have common zeros; if they had, we could simplify $d(z,w)$ by extracting those common zeros. In particular, we may assume that $\phi(0) \neq 0$.

(iii) As a consequence of (i) and (ii) we conclude that

$$In \left\{ d(Z,Z)R \right\} = In \left\{ R - FRF^* \right\}$$

where $F := \phi^{-1}(Z)\psi(Z)$, is a lower-triangular Toeplitz matrix whose first column is determined by the coefficients of the power-series expansion of $\psi(z)/\phi(z)$.

We have recently shown that our factorization procedure can be extended to matrices for which the generalized displacement $R - FRF^*$ has low rank, where F can be any lower triangular matrix; when F is a lower-triangular Toeplitz matrix the new procedure reduces to the one reported in [Lev-Ari and Kailath (1986)]. The efficiency of this new factorization procedure depends upon the structure of the matrix F, because the procedure involves repeated calculation of products between certain vectors and matrices of the form $(F - f_{i,i}I)(I - f_{i,i}^*F)^{-1}$, where $f_{i,i}$ denotes the (i,i) element of the matrix F. The computational complexity of such products depends upon the structure of F.

Very recently, Mr. J. Chun has shown how F matrices composed Z of different sizes can be nicely employed to obtain efficient algorithms for the least-squares solution of various special systems including, for example, triangular and orthogonal factorization of composite matrices such as $(T_1 T_2)$, $(T_1 T_2^{-1} T_3$, $(T^T T)^{-1} T^T$.

3. BEZOUTIANS AND EFFICIENT ZERO-LOCATION

One fascinating application of the notion of generalized displacement structure is the construction of *Bezoutian matrices*. Originally, such matrices were associated with stability (and zero-location) tests with respect to the unit circle and the imaginary axis. We have extended this notion to structured matrices whose generating function has the form

$$B(z,w) = \frac{p(z)p^*(w) - p^{\#}(z)[p^{\#}(w)]^*}{[1 \quad z]J_d[1 \quad w]^*} \tag{14a}$$

where

$$J_d := \begin{bmatrix} \alpha & \beta^* \\ \beta & \delta \end{bmatrix}, \tag{14b}$$

and the sharp (#) denotes a suitably defined polynomial transformation. The remarkable property of such matrices is that most of their elements vanish *except the elements in the leading $n \times n$ principal submatrix*, where $n := deg\ p(z)$. This submatrix, which we denote by **B**, has full rank (i.e., *rank* **B** $= n$) when $p(z)$ and $p^{\#}(z)$ are coprime. Furthermore, the inertia of the Bezoutian matrix **B** (i.e., the number of its positive, zero and negative eigenvalues) serves to locate the zeros of the polynomial $p(z)$ with respect to the following partition of the complex plane,

$$\Omega_+ = \{z \ ; \ d(z,z) > 0\}$$

$$\Omega_0 := \{z \; ; \; d(z,z) = 0\} \tag{15}$$
$$\Omega_- := \{z \; ; \; d(z,z) < 0\}$$

More specifically the inertia of **B** indicates how many zeros are shared by $p(z)$ and $p^{\#}(z)$ and how many of the remaining zeros are in Ω_+ and in Ω_-. Our fast factorization procedure makes it possible to determine the inertia of a Bezoutian matrix in $O(n^2)$ operations. For Bezoutians on the imaginary axis and the unit circle our formulation leads (among other possibilities) to the Routh-Hurwitz and Schur-Cohn tests, and serves to delimit the family of $O(n^2)$ polynomial zero-location procedures. Indeed, we conjecture that every $O(n^2)$ procedure corresponds to some specific choice in the four-dimensional "universe of choice" that we described in Section 2.

4. SINGULAR RECURSIONS

The fundamental recursion step (9) can be carried out if, and only if, the scalar $R_i(\zeta_i,\zeta_i)$ does not vanish. Thus, the recursion can be completed (for $i = 0,1,...,n$) if, and only if, the matrix **R** is *strongly regular*, i.e., all its leading principal minors do not vanish. Various solutions have been proposed, to overcome *recursion singularities* (i.e., the case when $R_i(\zeta_i,\zeta_i) = 0$), mostly involving some perturbation of the data or the recursion parameters. For instance, Vaidyanathan and Mitra (1987) suggest moving the point ζ_i , so that $R_i(\zeta_i,\zeta_i) \neq 0$ and the recursion can be carried on. This approach, in addition to being rather ad hoc, cannot be used in applications that involve *predetermined values* for $\{\zeta_i\}$, such as the commonly used choice $\zeta_i = 0$. There is a variety of similar approaches in the literature.

Our approach to overcome recursion singularities is different and is based on the

observation that a Hermitian matrix with singular leading principal minors has a *block-triangular* factorization of the form $R = L D L^*$ where L is lower triangular with unity diagonal elements and D is a *block-diagonal* matrix. The sizes of the blocks in D are determined by the *rank-profile* of R, i.e., by the variation of the nullity (=dimension of null space) of the leading principal submatrix $R_{0:n}$ as a function of n. This implies that whenever $R_i(\zeta_i, \zeta_i) = 0$ it may still be possible to "jump over" the singularity and compute $G_{i+\nu}(z)$ directly from $G_i(z)$ via a modified version of (9), where ν is the size of the block in D whose top left element occupies the (i,i) position in D.

We have recently derived this modified version of the fundamental recursion (9), for two particular examples: the Routh-Hurwitz algorithm and the Schur-Cohn algorithm. The former involves a generator of the form

$$\{d(z,w),\ G(z),\ J\} = \left\{ z + w^*,\ [p(z) + p^*(-z)\quad p(z) - p^*(-z)],\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

where $p(z)$ is a given polynomial (the recursion parameters are $\zeta_i = 0$, $\tau_i = j\infty$, $U_i = J$). The latter involves a generator of the form

$$\{d(z,w),\ G(z),\ J\} = \left\{ 1 - zw^*,\ [p(z) + p^\#(z)\quad p(z) - p^\#(z)],\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

where $p^\#(z) := z^{\deg p(z)} [p(1/z^*)]^*$ (and the recursion parameters are $\zeta_i = 0$, $\tau_i = 1$, $U_i = J$). Our modified recursion carries out the transformation $G_i(z) \rightarrow G_{i+\nu}(z)$ as a sequence of *single-step transformations*, viz.,

$$G_i(z) \rightarrow G_{i+1}(z) \rightarrow G_{i+2}(z) \rightarrow \cdots \rightarrow G_{i+\nu}(z)$$

where the intermediate generators $G_{i+k}(z)$ ($i \le k \le \nu - 1$), serve to determine the

triangular factor **L** of **R** *column by column*, rather than by block-columns (Pal and Kailath, (1988)).

5. RECURSIVE LAYER PEELING

The fundamental factorization procedure for structured matrices, viz.,

$$(z - \zeta_i)G_{i+1}(z) = G_i(z)\Theta_i(z)$$

is, at the same time, also a *layer-peeling*, procedure. Starting with $G_0(z)$, which we can interpret as boundary data for a layered medium, we identify an elementary layer with chain-scattering matrix $\Theta_0(z)$, then "peel" it off to obtain $G_1(z)$, the boundary data for the rest of the medium (with the first layer removed), and repeat the same procedure again and again. Such layer-peeling recursions have been used in cascade filter synthesis (see, e.g., Dewilde, Vieira and Kailath, (1978); Vaidyanathan and Mitra (1984)), in inverse scattering (Bruckstein and Kailath (1987)), zero-location (Lev-Ari, Bistritz and Kailath (1987)), and model-order reduction (see, e.g., Genin and Kung (1981)).

The classical work of Schur (1917) forms the basis for much of the subsequent work on layer peeling procedures. Schur's algorithm associates a sequence of so-called *reflection coefficients*, all with magnitude bounded by unity, with every passive scattering function, i.e., a function $S(z)$ that is analytic and bounded by 1 in the unit disc. In particular, if $S(z)$ is an *all-pass function*, which means that $|S(e^{j\theta})| = 1$ for all θ, then Schur's algorithm produces a *finite* sequence of reflection coefficients $\{k_i ; 0 \leq i \leq n\}$, where $|k_n| = 1$ and $|k_i| < 1$ for $0 \leq i \leq n - 1$. Another property of the algorithm is that starting with a passive

scattering function it generates a sequence of such functions. This is the essence of layer peeling: a single step of the Schur algorithm applied to a passive medium leaves a medium with the same property, which makes it possible to apply the same step again and again.

In addition to the recursive characterization of passivity via the constraint on the magnitude of the reflection coefficients, Schur also introduced an operator-norm characterization of passivity: he proved that for any function $f(z)$ that is analytic in the unit disc we can construct an infinite lower-triangular Toeplitz matrix whose first column consists of the coefficients of the power series expansion of $f(z)$, viz.,

$$\mathbf{L}(f) = \begin{bmatrix} f_0 & & & \\ f_1 & f_0 & & \mathbf{0} \\ f_2 & f_1 & \cdot & \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \end{bmatrix} \tag{16a}$$

such that

$$\sup_{|z| < 1} |f(z)| \le 1 \iff \|\mathbf{L}(f)\|_2 \le 1 \tag{16b}$$

where $\|A\|_2$ denotes the conventional (spectral) norm of a matrix A, i.e.,

$$\|A\|_2 := \sup_x \frac{\|Ax\|_2}{\|x\|_2} \tag{16c}$$

and $\|x\|_2$ denotes the Euclidean (l_2) norm of a vector x.

Schur's algorithm has been later applied also to functions with *poles in the unit disc*, but only to rational "all-pass" functions, i.e., to functions $f(z)$ of the form

$$f(z) = \lambda \frac{p^\#(z)}{p(z)} \tag{17a}$$

where $p^\#(z)$ denotes the conjugate reversal operation, viz.,

$$p^{\#}(z) = z^{deg\ p(z)}\left[p(1/z^*)\right]^* \tag{17b}$$

The well known Schur-Cohn test associates with each such function[†] a finite sequence of reflection coefficients, some of which have magnitudes larger than 1. Moreover, it follows from the properties of Bezoutians on the unit disc that the number of poles of $f(z) = p^{\#}(z)/p(z)$ inside the unit-disc equals the number of singular values of the matrix $\mathbf{L}(f)$ that are larger than $|\lambda|$ or, equivalently, the number of negative eigenvalues of the following finite rank matrix,

$$\mathbf{R} := |\lambda|^2 \mathbf{I} - \mathbf{L}(f)\mathbf{L}^*(f) \ . \tag{18}$$

It also follows that the so-called *Pick matrix*

$$\tilde{\mathbf{R}} := \left\{ \frac{|\lambda|^2 - f(\zeta_i)f^*(\zeta_j)}{1 - \zeta_i\zeta_j^*} \ ; \quad 0 \le i,j \le n \right\} \tag{19}$$

is *congruent* to $\mathbf{R}$ of (18), viz.,

$$\tilde{\mathbf{R}} = \begin{bmatrix} 1 & \zeta_0 & \zeta_0^2 & \cdot & \cdot & \cdot \\ 1 & \zeta_1 & \zeta_1^2 & \cdot & \cdot & \cdot \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & \zeta_n & \zeta_n^2 & \cdot & \cdot & \cdot \end{bmatrix} \mathbf{R} \begin{bmatrix} 1 & \zeta_0 & \zeta_0^2 & \cdot & \cdot & \cdot \\ 1 & \zeta_1 & \zeta_1^2 & \cdot & \cdot & \cdot \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & \zeta_n & \zeta_n^2 & \cdot & \cdot & \cdot \end{bmatrix}^*$$

and therefore has the same number of negative eigenvalues. The significance of these results has recently been recognized in the context of approximation and model-order reduction (see, e.g., Genin and Kung (1981)).

Rational allpass functions of a given degree k are members in the family H_k^∞, which consists of all functions with k poles (or less) inside the unit circle, and whose magnitude is bounded on the unit circle, i.e., $\sup_{|z|=1} |f(z)| < \infty$. It turns out that the

---

[†] Assuming $p(z)$ has no zeros at $z = 0$, and applying the algorithm to $f(z)/\lambda$.

Schur-Cohn algorithm does not map the family H_k^∞ into itself, except when $k = 0$. This means that this algorithm does not admit the same layer-peeling interpretation as the standard Schur algorithm. Nevertheless, we have found that it is possible to modify the Schur algorithm in such a way that the resulting recursion indeed maps the family H_k^∞ into itself and, therefore, admits the same layer-peeling interpretation as the classical Schur algorithm. In addition, it produces a transmission-line model that is similar to the one associated with the classical algorithm.

Furthermore, our modified recursion applies to every function $f(z) \in H_k^\infty$, and not just to allpass functions. Each layer in the resulting transmission line model has an indicator or 'sign'. While a *positive layer* maps H_k^∞ into itself, a *negative layer* maps H_k^∞ into H_{k-1}^∞ namely, it reduces by one the number of poles within the unit disc. Therefore, the number of negative layers in the transmission-line model that is generated by our modified algorithm equals the number of poles that the function $f(z)$ has within the unit disc. This is the key idea in the construction of efficient procedures for zero-location and model-order reduction.

This result should have some implications for control systems, where the underlying systems (linearized about an operating point) are often unstable. We are exploring these implications and attempting to relate them to results obtained in the last few years in the active field called H_∞-control.

REFERENCES

1. Y. Bistritz, H. Lev-Ari and T. Kailath, "Immittance-Domain Levinson Algorithms," *Proc. 1986 IEEE Int. Conf. Acoust. Speech and Signal Processing,* Tokyo, Japan, April 1986.

2. Y. Bistritz, H. Lev-Ari and T. Kailath, "Immittance-Domain Levinson Algorithms," submitted to the *IEEE Trans. Inform. Theory,* 1987.

3. A.M. Bruckstein and T. Kailath, "An Inverse Scattering Framework for Several Problems in Signal Processing," *IEEE ASSP Magazine,* Vol. 4, No. 1, pp. 6-20, Jan. 1987.

4. E.F. Deprettere and P. Dewilde, "Orthogonal Cascade Realization of Real Multiport Digital Filters," *Circuit Theory and Applications,* Vol. 8, pp. 245-272, 1980.

5. P. Dewilde, A. Vieira and T. Kailath, "On A Generalized Szegö-Levinson Realization Algorithm for Optimal Linear Predictors Based on a Network Synthesis Approach," *IEEE Trans. Circ. Syst.,* Vol. CAS-25, pp. 663-675, Sept. 1978.

6. Y. Genin and S.Y. Kung, "A Two Variable Approach to the Model Reduction Problem with Hankel Norm Criterion," *IEEE Trans. Circ. Syst.,* Vol. CAS-28, pp. 912-924, Sept. 1981.

7. G. Heinig and K. Rost, *Algebraic Methods for Toeplitz-like Matrices and Operators,* Akademie-Verlag, Berlin 1984.

8. G. Heinig and K. Rost, "Inversion of Generalized Toeplitz-plus-Hankel Matrices," *Wiss. Z. Techn. Hochsch. Karl-Marx-Stadt,* Vol. 29, pp. 209-211, 1987.

9. T. Kailath, S.-Y. Kung and M. Morf, "Displacement Ranks of Matrices and Linear Equations," *J. Math. Anal. and Appl.,* Vol. 68, No. 2, pp. 395-407, 1979. See also *Bull. Amer. Math. Soc.,* Vol. 1, pp. 769-773, Sept. 1979.

10. H. Lev-Ari, Research Notes, 1987.

11. H. Lev-Ari, Y. Bistritz and T. Kailath, "Generalized Bezoutians and Families of Efficient Root-Location Procedures," submitted to *IEEE Trans. Circ. Syst.,* 1987.

12. H. Lev-Ari and T. Kailath, "Lattice-Filter Parametrization and Modeling of Nonstationary Processes," *IEEE Trans. Inform. Thy.*, Vol. IT-30, pp. 2-16, Jan. 1984.

13. H. Lev-Ari and T. Kailath, "Triangular Factorization of Structured Hermitian Matrices," in *I. Schur Methods in Operator Theory and Signal Processing, Operator Theory: Advances and Applications*, I. Gohberg (ed.), Vol. 18, pp. 301-324, Birkhäuser Verlag, Basel, 1986.

14. H. Lev-Ari and T. Kailath, "Lossless Cascade Networks: The Crossroads of Stochastic Estimation, Inverse Scattering and Filter Synthesis," *Proc. 1987 Int. Symp. Circ. & Syst.*, pp. 1088-1091, Philadelphia, Pennsylvania, May 1987.

15 D. Pal and T. Kailath, "Inertia Extraction of Bezoutians and Root Location of Polynomials. Part I: Imaginary Axis Case; and Part II: Unit Circle Case," preprint, 1988.

16. S.K. Rao and T. Kailath, "Orthogonal Digital Filters for VLSI Implementation," *IEEE Trans. Circ. Syst.*, Vol. CAS-31, pp. 933-945, Nov. 1984.

17. S.K. Rao and T. Kailath, "VLSI Arrays for Digital Signal Processing: Part I. A Model Identification Approach to Digital Filter Realization," *IEEE Trans. Circ. Syst.*, Vol. CAS-31, Nov. 1985.

18. I. Schur, "Über Potenzreihen, die in Innern des Einheitskreises Beschränkt Sind," *Journal für die Reine und Angewandte Mathematik*, Vol. 147, pp. 205-232, Berlin, 1917, and Vol. 148, pp. 122-145, Berlin, 1918. English translation in *I. Schur Methods in Operator Theory and Signal Processing, Operator Theory: Advances and Applications*, I. Gohberg (ed.), Vol. 18, pp. 31-88, Birkhäuser Verlag, Basel, 1986.

19. P.P. Vaidyanathan and S.K. Mitra, "Low Passband Sensitivity Digital Filters: A Generalized Viewpoint and Synthesis Procedures," *Proc. IEEE*, Vol. 72, pp. 404-423, Apr. 1984.

20. P.P. Vaidyanathan and S.K. Mitra, "A Unified Structural Interpretation of Some Well-Known Stability-Test Procedures for Linear Systems," *Proc. IEEE*, Vol. 75, pp. 478-497, Apr. 1984.